Eur. Phys. J. C 47, 227–234 (2006)
Digital Object Identifier (DOI) 10.1140/epjc/s2006-02540-8

THE EUROPEAN
PHYSICAL JOURNAL C

Unique nilpotent symmetry transformations for matter fields in QED: augmented superfield formalism

R.P. Malik[1,2,a]

[1] S.N. Bose National Centre for Basic Sciences, Block JD, Sector III, Salt Lake, Calcutta 700 098, India
[2] Centre of Advanced Studies, Physics Department, Banaras Hindu University, Varanasi 221 005, India

Received: 17 January 2006 / Revised version: 5 March 2006 /
Published online: 19 April 2006 –

Abstract. We derive the off-shell nilpotent (anti-)BRST symmetry transformations for the interacting $U(1)$ gauge theory of quantum electrodynamics (QED) in the framework of the augmented superfield approach to the BRST formalism. In addition to the horizontality condition, we invoke another gauge invariant condition on the six $(4,2)$-dimensional supermanifold to obtain the exact and unique nilpotent symmetry transformations for all the basic fields present in the (anti-)BRST invariant Lagrangian density of the physical four $(3+1)$-dimensional QED. The above supermanifold is parametrized by four even space–time variables x^μ (with $\mu = 0,1,2,3$) and two odd variables (θ and $\bar{\theta}$) of the Grassmann algebra. The new gauge invariant condition on the supermanifold owes its origin to the (super) covariant derivatives and leads to the derivation of unique nilpotent symmetry transformations for the matter fields. The geometrical interpretations for all the above off-shell nilpotent (anti-)BRST transformations are also discussed.

PACS. 11.15.-q; 12.20.-m; 03.70.+k

1 Introduction

The usual superfield approach [3–5, 12] to the Becchi–Rouet–Stora–Tyutin (BRST) formalism (see e.g. [6–9] for details) for a p-form (with $p = 1,2,3,\dots$) Abelian gauge theory delves deep into the geometrical aspects of the nilpotent (anti-)BRST symmetries (and corresponding nilpotent generators) for the p-form gauge fields and the underlying (anti-)ghost fields of the theory. To be precise, under the above approach, the D-dimensional gauge theory is first considered on the $(D,2)$-dimensional supermanifold which is parametrized by the D-number of even space–time commuting coordinates x^μ (with $\mu = 0,1,2,\dots,D-1$) and two anticommuting (i.e. $\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$) odd variables ($\theta$ and $\bar{\theta}$) of the Grassmann algebra. After this, a $(p+1)$-form super curvature $\tilde{F}^{(p+1)} = \tilde{d}\tilde{A}^{(p)}$ is constructed from (i) the super exterior derivative $\tilde{d} = dx^\mu\partial_\mu + d\theta\partial_\theta + d\bar{\theta}\partial_{\bar{\theta}}$ (with $\tilde{d}^2 = 0$) and (ii) the super p-form connection $\tilde{A}^{(p)}$ on the $(D,2)$-dimensional supermanifold. Subsequently, this super curvature $\tilde{F}^{(p+1)}$ is equated, due to the so-called horizontality condition [1–5], with the ordinary $(p+1)$-form curvature $F^{(p+1)} = dA^{(p)}$ constructed by the ordinary D-dimensional exterior derivative $d = dx^\mu\partial_\mu$ (with $d^2 = 0$) and the ordinary p-form connection $A^{(p)}$ defined on the ordinary D-dimensional Minkowskian flat space–time manifold on which the starting p-form gauge theory (endowed with the first-class constraints) exists.

The above horizontality condition is christened as the soul-flatness condition in [6], which mathematically amounts to setting equal to zero all the Grassmannian components of the (anti-)symmetric tensor that defines the $(p+1)$-form super curvature $\tilde{F}^{(p+1)}$ on the $(D,2)$-dimensional supermanifold. The process of reduction of the $(D,2)$-dimensional super curvature to the D-dimensional ordinary curvature (i.e. the equality $\tilde{F}^{(p+1)} = F^{(p+1)}$) leads to the derivation of the nilpotent (anti-)BRST symmetry transformation for the p-form gauge fields and the (anti-)commuting (anti-)ghost fields of the theory[1]. As a bonus and by-product, the geometrical interpretations for the nilpotency and anticommutativity properties of the conserved and nilpotent (anti-)BRST charges[2] emerge very naturally on the supermanifold. However, these beautiful connections between the geometrical aspects of the

[1] It can be seen that, for the 2-form Abelian gauge theory, the bosonic and fermionic ghosts do exist in the BRST formalism and their transformations can be derived using the usual superfield formalism [10].

[2] These charges turn out to be the translational generators along the Grassmannian directions of the supermanifold. Their nilpotency and anticommutativity properties are also found to be encoded in the specific properties associated with the above translational generators (see e.g. [11–16] for details).

[a] e-mail: malik@bhu.ac.in

supermanifold and the (anti-)BRST transformations (and their corresponding generators) remain absolutely confined to the gauge fields and the (anti-)ghost fields. The above usual superfield formalism [1–5, 10] does not shed any light on the (anti-)BRST symmetry transformations associated with the matter fields of an interacting Abelian gauge theory.

It is worthwhile to mention, at this juncture, that the usual superfield approach has also been applied to the case of four $(3+1)$-dimensional (4D) 1-form ($A^{(1)} = dx^\mu A_\mu$) non-Abelian gauge theory where the 2-form super curvature $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + \tilde{A}^{(1)} \wedge \tilde{A}^{(1)}$, defined on the six $(4, 2)$-dimensional supermanifold, is equated to the 4D ordinary 2-form curvature $F^{(2)} = dA^{(1)} + A^{(1)} \wedge A^{(1)}$ (constructed from $d = dx^\mu \partial_\mu$ and $A^{(1)}$) due to the horizontality condition. As expected, this procedure of covariant reduction of the 2-form super curvature to ordinary curvature leads to the derivation of nilpotent (anti-)BRST symmetry transformations, associated with the non-Abelian gauge field and the anticommuting (anti-)ghost fields of the theory (see e.g. [4]). The matter (Dirac) fields of the interacting non-Abelian gauge theory are found to play no role in the above covariant reduction associated with the horizontality condition. As a consequence, one does not obtain the nilpotent (anti-)BRST symmetry transformations for the matter (Dirac) fields by the usual superfield formalism.

The purpose of our present paper is to derive uniquely and exactly the off-shell nilpotent (anti-)BRST symmetry transformations for the matter (Dirac) fields of quantum electrodynamics (QED) in 4D by invoking a new gauge invariant restriction, besides the usual horizontality condition, on the six $(4, 2)$-dimensional supermanifold. In this context, it is worthwhile to point out that, in a recent set of papers [11–16], the usual superfield approach (with horizontality condition alone) has been extended to include the invariance of the conserved (matter) currents/charges to obtain all the nilpotent and anticommuting (anti-)BRST symmetry transformations for all the basic fields of the interacting (non-)Abelian gauge theories as well as the reparametrization-invariant free scalar and spinning relativistic particle(s). However, the nilpotent (anti-)BRST symmetry transformations, which emerge due to the latter restrictions on the supermanifold, were not proved to be mathematically unique. One of the central themes of our present paper is to demonstrate that the new gauge invariant restriction on the supermanifold (cf. (11)), which owes its origin to the (super) covariant derivatives, leads to the derivation of the off-shell nilpotent (anti-)BRST symmetry transformations for the matter fields uniquely. It is very gratifying that the geometrical interpretations for the (anti-)BRST symmetry transformations and their corresponding nilpotent generators (that emerge especially after the application of the horizontality condition) remain intact under the above extended version of the usual superfield approach to the BRST formalism. Thus, there is very neat mutual consistency, conformity and complementarity between the above two restrictions on the supermanifold. We christen our present approach as well as that of [11–16] as the augmented superfield formalism because (i) all these attempts are a set of consistent extensions (and, in some sense, generalizations) of the usual superfield approach to the BRST formalism and (ii) the nilpotent and anticommuting (anti-)BRST transformations for all the fields of the 4D interacting 1-form Abelian $U(1)$ gauge theory are derived in this superfield approach to the BRST formalism.

The contents of our present paper are organized as follows. In Sect. 2, we recapitulate the bare essentials of the off-shell nilpotent (anti-)BRST symmetry transformations in the Lagrangian formulation for the 4D interacting $U(1)$ gauge theory (i.e. QED). For the paper to be self-contained, Sect. 3 is devoted to a brief description of the derivation of the above symmetry transformations for the gauge field A_μ and the (anti-)ghost fields $(\bar{C})C$ by exploiting the usual horizontality condition on the six $(4, 2)$-dimensional supermanifold [4, 11, 12]. The central result of our paper is contained in Sect. 4, where we derive uniquely the off-shell nilpotent symmetry transformations for the matter (Dirac) fields in the framework of the augmented superfield formalism by exploiting a gauge invariant restriction on the six $(4, 2)$-dimensional supermanifold. Its gauge covariant version does not lead to the derivation of correct nilpotent symmetries (see e.g. the appendix). Finally, we make some concluding remarks and pinpoint a few future directions for further investigations in Sect. 5.

2 Preliminary: nilpotent (anti-)BRST symmetries

To set the notation and conventions that will be useful for our later discussions, we begin with the (anti-)BRST invariant Lagrangian density $\mathcal{L}_B$ for the interacting four $(3+1)$-dimensional (4D) $U(1)$ gauge theory (i.e. QED) in the Feynman gauge[3] [6–9]:

$$\mathcal{L}_B = -\frac{1}{4}\, F^{\mu\nu} F_{\mu\nu} + \bar{\psi}\,(\mathrm{i}\gamma^\mu D_\mu - m)\,\psi + B\,(\partial \cdot A) + \frac{1}{2}\, B^2 - \mathrm{i}\, \partial_\mu \bar{C} \partial^\mu C\,, \qquad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor for the $U(1)$ gauge theory and the covariant derivative on the matter (Dirac) field $D_\mu \psi = \partial_\mu \psi + \mathrm{i}e A_\mu \psi$ leads to the interaction term between the $U(1)$ gauge field A_μ and Dirac fields ψ and $\bar{\psi}$ of mass m and electric charge e

[3] We adopt here the conventions and notation such that the 4D flat Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1)$ and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2, F_{0i} = E_i = \partial_0 A_i - \partial_i A_0 = F^{i0}, F_{ij} = -\varepsilon_{ijk}\, B_k, B_i = -\frac{1}{2}\varepsilon_{ijk} F_{jk}, \partial^\mu A_\mu = \partial_\mu A^\mu = (\partial \cdot A) \equiv \partial_0 A_0 - \partial_i A_i$. Here E_i and B_i are the electric and magnetic fields and ε_{ijk} is the 3D totally antisymmetric Levi–Civita tensor in the space indices. Furthermore, the Greek indices $\mu, \nu, \ldots = 0, 1, 2, 3$ correspond to the space–time directions on the 4D Minkowskian space–time manifold and the Latin indices $i, j, k, \ldots = 1, 2, 3$ stand for the space directions only.

(i.e. $-e\bar{\psi}\gamma^\mu A_\mu \psi$). In fact, this term arises through the term $i\bar{\psi}\gamma^\mu D_\mu \psi$ that is present in the Lagrangian density (1), where γ^μ's are the 4×4 Dirac matrices. The anticommuting ($C\bar{C}+\bar{C}C=0, C^2=\bar{C}^2=0, C\psi+\psi C=0$, etc.) (anti-)ghost fields $(\bar{C})C$ are required to maintain the unitarity and 'quantum' gauge (i.e. BRST) invariance together at any arbitrary order of perturbation theory[4]. The Nakanishi–Lautrup auxiliary field B is required to linearize the gauge-fixing term $-\frac{1}{2}(\partial\cdot A)^2$ in (1). The above Lagrangian density (2) respects the following local, covariant, continuous, off-shell nilpotent ($s_{(a)b}^2=0$) and anticommuting ($s_b s_{ab}+s_{ab}s_b=0$) (anti-)BRST ($s_{(a)b}$)[5] symmetry transformations (see e.g. [6–9] for all the details)

$$\begin{aligned}
&s_b A_\mu = \partial_\mu C\,, && s_b C = 0\,,\\
&s_b \bar{C} = \mathrm{i}B\,, && s_b\psi = -\mathrm{i}eC\psi\,,\\
&s_b\bar{\psi} = -\mathrm{i}e\bar{\psi}C\,, && s_b B = 0\,,\\
&s_b F_{\mu\nu} = 0\,, && s_b(\partial\cdot A) = \Box C\,,\\
&s_{ab}A_\mu = \partial_\mu\bar{C}\,, && s_{ab}\bar{C} = 0\,,\\
&s_{ab}C = -\mathrm{i}B\,, && s_{ab}\psi = -\mathrm{i}e\bar{C}\psi\,,\\
&s_{ab}\bar{\psi} = -\mathrm{i}e\bar{\psi}\bar{C}\,, && s_{ab}B = 0\,,\\
&s_{ab}F_{\mu\nu} = 0\,, && s_{ab}(\partial\cdot A) = \Box\bar{C}\,.
\end{aligned}\qquad (2)$$

The noteworthy points, at this stage, are (i) under the (anti-)BRST transformations, it is the kinetic energy term (more precisely $F_{\mu\nu}$ itself) that remains invariant and the gauge-fixing term $(\partial\cdot A)$ transforms. It should be emphasized that the antisymmetric field strength tensor $F_{\mu\nu}$ remains invariant under the original local gauge transformation (i.e. $\delta_g A_\mu = \partial_\mu\alpha(x)$ with $\alpha(x)$ as an infinitesimal gauge parameter), too. In fact, all the gauge invariant quantities remain invariant quantities under the (anti-)BRST transformations as well. (ii) The starting local $U(1)$ gauge invariant theory is endowed with the first-class constraints in the language of Dirac's classification scheme for constraints. These constraints are found to be encoded in the physicality criteria where physical states ($|\text{phys}>$) (of the total Hilbert space of quantum states) are annihilated (i.e. $Q_{(a)b}|\text{phys}>=0$) by the conserved and nilpotent (anti-)BRST charges $Q_{(a)b}$ (see e.g. [6–9] for details). (iii) The local, conserved and nilpotent charges $Q_{(a)b}$ can be computed by exploiting the Noether theorem. These charges do generate the nilpotent and anticommuting (anti-)BRST transformations. This statement can be succinctly expressed in mathematical form as given below:

$$s_r\Omega(x) = -\mathrm{i}\,[\Omega(x), Q_r]_\pm\,,\qquad r = b, ab\,,\qquad (3)$$

where the local generic field $\Omega = A_\mu, C, \bar{C}, \psi, \bar{\psi}, B$ and the $(+)-$ signs, as the subscripts on the square bracket, stand for the bracket to be an (anti-)commutator for the local generic field Ω being (fermionic) bosonic in nature.

3 Symmetries for gauge and (anti-)ghost fields: usual superfield formalism

To obtain the off-shell nilpotent symmetry transformations for the gauge field A_μ and the (anti-)ghost fields $(\bar{C})C$, we begin with a six $(4,2)$-dimensional supermanifold parametrized by the superspace coordinates $Z^M = (x^\mu, \theta, \bar{\theta})$, where x^μ ($\mu = 0,1,2,3$) are the even (bosonic) space–time coordinates and θ and $\bar{\theta}$ are the two odd (Grassmannian) coordinates. On this supermanifold, one can define a 1-form super connection $\tilde{A}^{(1)} = dZ^M\tilde{A}_M$, where $\tilde{A}_M = (B_\mu(x,\theta,\bar{\theta}), \mathcal{F}(x,\theta,\bar{\theta}), \bar{\mathcal{F}}(x,\theta,\bar{\theta}))$ are the component multiplet superfields [4, 3] with B_μ being bosonic in nature and $\mathcal{F}, \bar{\mathcal{F}}$ being fermionic (i.e. $\mathcal{F}^2 = \bar{\mathcal{F}}^2 = 0$). The superfields $B_\mu(x,\theta,\bar{\theta}), \mathcal{F}(x,\theta,\bar{\theta}), \bar{\mathcal{F}}(x,\theta,\bar{\theta})$ can be expanded in terms of the basic fields $A_\mu(x), C(x), \bar{C}(x)$, the auxiliary field $B(x)$ of (1) and some extra fields as [4, 3, 11]

$$\begin{aligned}
B_\mu(x,\theta,\bar{\theta}) &= A_\mu(x) + \theta\,\bar{R}_\mu(x) + \bar{\theta}\,R_\mu(x) + \mathrm{i}\,\theta\,\bar{\theta}S_\mu(x)\,,\\
\mathcal{F}(x,\theta,\bar{\theta}) &= C(x) + \mathrm{i}\,\theta\bar{B}(x) + \mathrm{i}\,\bar{\theta}\,\mathcal{B}(x) + \mathrm{i}\,\theta\,\bar{\theta}\,s(x)\,,\\
\bar{\mathcal{F}}(x,\theta,\bar{\theta}) &= \bar{C}(x) + \mathrm{i}\,\theta\,\bar{\mathcal{B}}(x) + \mathrm{i}\,\bar{\theta}\,B(x) + \mathrm{i}\,\theta\,\bar{\theta}\,\bar{s}(x)\,.
\end{aligned}\qquad (4)$$

It is straightforward to note that the local fields $R_\mu(x)$, $\bar{R}_\mu(x)$, $C(x)$, $\bar{C}(x)$, $s(x)$ and $\bar{s}(x)$ are fermionic (anticommuting) in nature and $A_\mu(x)$, $S_\mu(x)$, $\mathcal{B}(x)$, $\bar{\mathcal{B}}(x)$, $B(x)$ and $\bar{B}(x)$ are bosonic (i.e. commuting) so that, in the above expansion, the bosonic and fermionic degrees of freedom match. This requirement is essential for the sanctity of any arbitrary supersymmetric theory described in the framework of the superfield formulation. In fact, all the secondary fields will be expressed in terms of basic fields (and their derivatives) due to the restrictions emerging from the application of the horizontality condition, namely

$$\begin{aligned}
\frac{1}{2}\,(dZ^M\wedge dZ^N)\,\tilde{F}_{MN} &= \tilde{d}\tilde{A}^{(1)} \equiv dA^{(1)}\\
&= \frac{1}{2}(dx^\mu\wedge dx^\nu)\,F_{\mu\nu}\,,
\end{aligned}\qquad (5)$$

where the super exterior derivative $\tilde{d}$ and the connection super 1-form $\tilde{A}^{(1)}$ are defined as

$$\begin{aligned}
\tilde{d} &= dZ^M\partial_M\\
&= dx^\mu\partial_\mu + d\theta\partial_\theta + d\bar{\theta}\partial_{\bar{\theta}}\,,\\
\tilde{A}^{(1)} &= dZ^M\tilde{A}_M\\
&= dx^\mu B_\mu(x,\theta,\bar{\theta}) + d\theta\bar{\mathcal{F}}(x,\theta,\bar{\theta}) + d\bar{\theta}\mathcal{F}(x,\theta,\bar{\theta})\,.
\end{aligned}\qquad (6)$$

[4] The full strength of the (anti-)ghost fields turns up in the discussion of the unitarity and gauge invariance for the perturbative computations in the realm of non-Abelian gauge theory where the loop diagrams of the gauge (gluon) fields play a very important role. In fact, for each such gluon loop diagram, a ghost loop diagram is required for the precise proof of unitarity in the theory (see e.g. [7, 17]).

[5] We adopt here the notation and conventions followed in [8, 9]. In fact, in its full blaze of glory, a nilpotent ($\delta_B^2=0$) BRST transformation δ_B is equal to the product of an anticommuting ($\eta C=-C\eta, \eta\bar{C}=-\bar{C}\eta, \eta\psi=-\psi\eta, \eta\bar{\psi}=-\bar{\psi}\eta$, etc.) space–time-independent parameter η and s_b (i.e. $\delta_B = \eta\,s_b$) with $s_b^2=0$.

In physical language, this requirement (i.e. (5)) implies that the physical fields E_i and B_i, derived from the curvature term $F_{\mu\nu}$, do not have any contribution from the Grassmannian variables. In other words, the physical electric and magnetic fields (E_i and B_i for the 4D QED) remain unchanged in the superfield formulation. Mathematically, the condition (5) implies the 'flatness' of all the components of the super curvature tensor $\tilde{F}_{MN}$ (derived from the super 2-form) that are directed along the θ and/or $\bar{\theta}$ directions of the supermanifold. To this end in mind, let us first expand $\tilde{d}\tilde{A}^{(1)}$ explicitly as

$$\begin{aligned}\tilde{d}\tilde{A}^{(1)} &= (dx^\mu \wedge dx^\nu)(\partial_\mu B_\nu) - (d\theta \wedge d\theta)\ \left(\partial_\theta \bar{\mathcal{F}}\right) \\ &+ (dx^\mu \wedge d\bar{\theta})(\partial_\mu \mathcal{F} - \partial_{\bar{\theta}} B_\mu) \\ &- (d\theta \wedge d\bar{\theta})\left(\partial_\theta \mathcal{F} + \partial_{\bar{\theta}} \bar{\mathcal{F}}\right) + (dx^\mu \wedge d\theta) \\ &\times \left(\partial_\mu \bar{\mathcal{F}} - \partial_\theta B_\mu\right) - (d\bar{\theta} \wedge d\bar{\theta})\left(\partial_{\bar{\theta}} \mathcal{F}\right) . \end{aligned} \tag{7}$$

Ultimately, the application of the soul-flatness (horizontality) condition ($\tilde{d}\tilde{A}^{(1)} = dA^{(1)}$) leads to the following relationships between extra secondary fields and basic fields (and their derivatives) (see e.g. [11, 12] for all the details):

$$\begin{aligned} &R_\mu\,(x) = \partial_\mu\, C(x)\,, && \bar{R}_\mu\,(x) = \partial_\mu \bar{C}(x)\,, \\ &s\,(x) = \bar{s}\,(x) = 0\,, && S_\mu\,(x) = \partial_\mu B\,(x)\,, \\ &B\,(x) + \bar{B}\,(x) = 0\,, && \mathcal{B}\,(x) = \bar{\mathcal{B}}(x) = 0\,. \end{aligned} \tag{8}$$

The insertion of all the above values in the expansion (4) leads to

$$\begin{aligned} B^{(h)}_\mu\,(x,\theta,\bar{\theta}) &= A_\mu(x) + \,\theta\,\partial_\mu \bar{C}(x) + \,\bar{\theta}\,\partial_\mu C(x) \\ &+ \mathrm{i}\,\theta\,\bar{\theta}\,\partial_\mu B(x)\,, \\ \mathcal{F}^{(h)}\,(x,\theta,\bar{\theta}) &= C(x)\ - \mathrm{i}\,\theta\,B(x)\,, \\ \bar{\mathcal{F}}^{(h)}\,(x,\theta,\bar{\theta}) &= \bar{C}(x) + \mathrm{i}\,\bar{\theta}\,B(x)\,, \end{aligned} \tag{9}$$

where our starting super expansion for the multiplet superfields of (4) has changed to $B_\mu \to B^{(h)}_\mu, \mathcal{F} \to \mathcal{F}^{(h)}, \bar{\mathcal{F}} \to \bar{\mathcal{F}}^{(h)}$ after the application of the horizontality condition. As a result, $\tilde{A}^{(1)} \to \tilde{A}^{(1)}_{(h)} = dx^\mu B^{(h)}_\mu + d\theta \bar{\mathcal{F}}^{(h)} + d\bar{\theta}\mathcal{F}^{(h)}$ in (6). In fact, the above reduction leads to the derivation of the (anti-)BRST symmetries for the gauge and (anti-)ghost fields of the Abelian gauge theory. In addition, this exercise provides the physical interpretation for the (anti-)BRST charges $Q_{(a)b}$ as the generators (cf. (3)) of translations (i.e. $(\mathrm{Lim}_{\bar{\theta}\to 0}(\partial/\partial\theta), \mathrm{Lim}_{\theta\to 0}(\partial/\partial\bar{\theta}))$ along the Grassmannian directions of the supermanifold. Both these observations can be succinctly expressed, in a combined way, by rewriting the super expansion (3.6) as

$$\begin{aligned} B^{(h)}_\mu\,(x,\theta,\bar{\theta}) &= A_\mu(x) + \,\theta\,(s_{ab} A_\mu(x)) + \,\bar{\theta}\,(s_b A_\mu(x)) \\ &+ \,\theta\,\bar{\theta}\,(s_b s_{ab} A_\mu(x))\,, \\ \mathcal{F}^{(h)}\,(x,\theta,\bar{\theta}) &= C(x)\ + \theta\,(s_{ab} C(x))\ + \bar{\theta}\,(s_b C(x)) \\ &+ \,\theta\,\bar{\theta}\,(s_b\, s_{ab} C(x))\,, \\ \bar{\mathcal{F}}^{(h)}\,(x,\theta,\bar{\theta}) &= \bar{C}(x) + \,\theta\,\left(s_{ab} \bar{C}(x)\right)\ + \bar{\theta}\,(s_b \bar{C}(x)) \\ &+ \,\theta\,\bar{\theta}\,\left(s_b\, s_{ab} \bar{C}(x)\right)\,, \end{aligned} \tag{10}$$

where it is evident that $s_b C = 0$ and $s_{ab}\bar{C} = 0$ have been taken into account. In fact, it is because of these inputs that the above expansion appears so symmetrical when expressed in terms of the (anti-)BRST transformations $s_{(a)b}$.

4 Symmetries for the Dirac fields: augmented superfield approach

It is obvious from the definition and property associated with the covariant derivative $D_\mu\psi(x) = \partial_\mu\psi(x) + \mathrm{i}eA_\mu(x)\psi(x)$ that an interesting combination of the $U(1)$ gauge field A_μ and the matter (Dirac) fields, through this derivative (i.e. $\bar{\psi}(x)D_\mu\psi(x)$), is a gauge (and, therefore, BRST) invariant quantity. In what follows, we shall derive the exact and unique expressions for the nilpotent symmetry transformations (2) for the matter fields by demanding the invariance of this gauge invariant quantity on the supermanifold. This statement can be mathematically expressed by the following equation:

$$\begin{aligned} &\bar{\Psi}(x,\theta,\bar{\theta})\ \left(\tilde{d} + \mathrm{i}\,e\,\tilde{A}^{(1)}_{(h)}\right)\,\Psi(x,\theta,\bar{\theta}) \\ &\quad = \bar{\psi}(x)\ \left(d + \mathrm{i}\,e\,A^{(1)}\right)\,\psi(x)\,, \end{aligned} \tag{11}$$

where $\tilde{d}$ and $\tilde{A}^{(1)}$ are the super exterior derivative and super 1-form connection (cf. (6)) on a six $(4,2)$-dimensional supermanifold and $d = dx^\mu\partial_\mu$ and $A^{(1)} = dx^\mu A_\mu$ are their counterparts on the ordinary 4D Minkowskian space–time manifold. In particular, $\tilde{A}^{(1)}_{(h)} = dx^\mu B^{(h)}_\mu + d\theta\bar{\mathcal{F}}^{(h)} + d\bar{\theta}\mathcal{F}^{(h)}$ is the expression for $\tilde{A}^{(1)}$ after the application of the horizontality condition (cf. (9)). The general super expansion of the superfields $(\Psi,\bar{\Psi})(x,\theta,\bar{\theta})$, corresponding to the ordinary Dirac fields $(\psi,\bar{\psi})(x)$, is taken as follows:

$$\begin{aligned} \Psi(x,\theta,\bar{\theta}) &= \psi(x) + i\,\theta\,\bar{b}_1(x) + \mathrm{i}\,\bar{\theta}\,b_2(x) + i\,\theta\,\bar{\theta}\,f(x)\,, \\ \bar{\Psi}(x,\theta,\bar{\theta}) &= \bar{\psi}(x) + \mathrm{i}\,\theta\,\bar{b}_2(x) + \mathrm{i}\,\bar{\theta}\,b_1(x) + \mathrm{i}\,\theta\,\bar{\theta}\,\bar{f}(x)\,. \end{aligned} \tag{12}$$

It is clear that, in the limit $(\theta,\bar{\theta}) \to 0$, we get back the Dirac fields $(\psi,\bar{\psi})$ of the Lagrangian density (1). Furthermore, the number of bosonic fields $(b_1,\bar{b}_1,b_2,\bar{b}_2)$ matches that of the fermionic fields $(\psi,\bar{\psi},f,\bar{f})$, so that the above expansion is consistent with the basic tenets of supersymmetry.

It is straightforward to see that there is only one term on the r.h.s. of (11), which can be explicitly expressed as $dx^\mu\ [\bar{\psi}(x)\ (\partial_\mu + \mathrm{i}eA_\mu(x))\ \psi(x)]$. However, it is evident that on the l.h.s. we shall have the coefficients of the differentials $dx^\mu, d\theta$ and $d\bar{\theta}$. To compute explicitly these coefficients, let us first focus on the first term of the l.h.s. of (11):

$$\begin{aligned} &\bar{\Psi}(x,\theta,\bar{\theta})\ \tilde{d}\,\Psi(x,\theta,\bar{\theta}) \\ &\quad = \bar{\Psi}\ (dx^\mu\ \partial_\mu)\ \Psi + \bar{\Psi}\ (d\theta\ \partial_\theta)\ \Psi + \bar{\Psi}\ (d\bar{\theta}\ \partial_{\bar{\theta}})\ \Psi\,, \end{aligned} \tag{13}$$

where it can be readily checked that $\partial_\theta\Psi = \mathrm{i}\bar{b}_1 + \mathrm{i}\bar{\theta}f$ and $\partial_{\bar{\theta}}\Psi = \mathrm{i}b_2 - \mathrm{i}\theta f$. Taking the help of the anticommuting properties of the Grassmannian variables and their differentials, we obtain the following explicit expressions for the coefficients of $d\theta$ and $d\bar\theta$ from (13):

$$-(d\theta)\Big[\,\mathrm{i}\,\bar\psi\,\bar b_1 - \mathrm{i}\,\bar\theta\,\big(\bar\psi\,f - \mathrm{i}\,b_1\,\bar b_1\big) - \theta\,\big(\bar b_2\,\bar b_1\big) - \theta\,\bar\theta\,\big(\bar b_2\,f + \bar f\,\bar b_1\big)\Big] - (d\bar\theta)\Big\{\,\mathrm{i}\,\bar\psi\,b_2 + \mathrm{i}\,\theta\,\big(\bar\psi\,f + \mathrm{i}\,\bar b_2\,b_2\big) - \bar\theta\,\big(b_1\,b_2\big) - \theta\,\bar\theta\,\big(b_1\,f + \bar f\,b_2\big)\Big\}, \tag{14}$$

where we have exploited $d\theta\,\bar\Psi = -\bar\Psi\,d\theta$, etc., and the expansion of $(\Psi,\bar\Psi)(x,\theta,\bar\theta)$ from (12). Using, once again, the anticommuting properties (i.e. $\theta^2 = \bar\theta^2 = 0, \theta\bar\theta + \bar\theta\theta = 0$) of the Grassmannian variables θ and $\bar\theta$, we obtain the following explicit expression for the coefficient $K_\mu(x,\theta,\bar\theta)$ of dx^μ from the first term of (13), namely

$$(dx^\mu)\,K_\mu(x,\theta,\bar\theta) \equiv (dx^\mu)\Big[\,\{\bar\psi\partial_\mu\psi\}(x) + \mathrm{i}\,\theta\,L_\mu(x) + \mathrm{i}\,\bar\theta\,M_\mu(x) + \mathrm{i}\,\theta\,\bar\theta\,N_\mu(x)\Big], \tag{15}$$

where the long-hand expressions for $L_\mu(x), M_\mu(x)$ and $N_\mu(x)$ are

$$\begin{aligned} L_\mu(x) &= \bar b_2\,\partial_\mu\,\psi - \bar\psi\,\partial_\mu\,\bar b_1\,,\\ M_\mu(x) &= b_1\,\partial_\mu\,\psi - \bar\psi\,\partial_\mu\,b_2\,,\\ N_\mu(x) &= \bar\psi\,\partial_\mu\,f + \bar f\,\partial_\mu\psi + \mathrm{i}\,\left(\bar b_2\,\partial_\mu\,b_2 - b_1\,\partial_\mu\,\bar b_1\right)\,. \end{aligned} \tag{16}$$

Let us concentrate on the explicit computation of the coefficients of $dx^\mu, d\theta$ and $d\bar\theta$ that emerge from the second term of the l.h.s. of (11) (i.e. $\mathrm{i}\,e\,\bar\Psi\,\tilde A^{(1)}_{(h)}\,\Psi$). This term, written in the component fields, has the following clear expansion:

$$\begin{aligned} &\mathrm{i}\,e\,\bar\Psi(x,\theta,\bar\theta)\,\tilde A^{(1)}_{(h)}\,\Psi(x,\theta,\bar\theta)\\ &= \mathrm{i}\,e\Big[\bar\Psi\,\Big(dx^\mu\,B^{(h)}_\mu\Big)\,\Psi + \bar\Psi\,\Big(d\theta\bar{\mathcal F}^{(h)}\Big)\,\Psi + \bar\Psi\,\Big(d\bar\theta\mathcal F^{(h)}\Big)\,\Psi\Big], \end{aligned} \tag{17}$$

where we shall be using the neat expressions for the expansions (9) obtained after the application of the horizontality condition. It is clear that the latter two terms of (17) lead to the computation of the coefficients of $d\theta$ and $d\bar\theta$. These are as quoted below:

$$\begin{aligned} &-\mathrm{i}\,e\,d\theta\,\Big(\bar\psi\bar C\psi\Big) + e\,d\theta\,\bar\theta\,\Big(\bar\psi\bar C b_2 - \bar\psi B\psi + b_1\bar C\psi\Big)\\ &+ e\,d\theta\,\theta\,\Big(\bar\psi\bar C\bar b_1 + \bar b_2\bar C\psi\Big) + e\,d\theta\,\theta\,\bar\theta\\ &\times\Big[\bar\psi\bar C f + \bar f\bar C\psi + \mathrm{i}\,\Big(b_1\bar C\bar b_1 + \bar b_2 B\psi - \bar b_2\bar C b_2 - \bar\psi B\bar b_1\Big)\Big],\\ &-\mathrm{i}\,e\,d\bar\theta\,\Big(\bar\psi C\psi\Big) + e\,d\bar\theta\,\theta\,\Big(\bar\psi C\bar b_1 + \bar\psi B\psi + \bar b_2 C\psi\Big)\\ &+ e\,d\bar\theta\,\bar\theta\,\Big(\bar\psi C b_2 + b_1 C\psi\Big),\\ &+ e\,d\bar\theta\,\theta\,\bar\theta\,\Big[\,\bar\psi C f + \bar f C\psi + \mathrm{i}\\ &\times\Big(b_1 C\bar b_1 + b_1 B\psi - \bar b_2 C b_2 - \bar\psi B b_2\Big)\Big]. \end{aligned} \tag{18}$$

The space–time component (i.e. the coefficient $E_\mu(x,\theta,\bar\theta)$ of dx^μ) that emerges from the expansion of the first term of (17) is given below:

$$(dx^\mu)E_\mu(x,\theta,\bar\theta) = (dx^\mu)\Big[\,\mathrm{i}\,e\,\{\bar\psi A_\mu\psi\}(x) + \theta\,F_\mu(x) + \bar\theta\,G_\mu(x) + \theta\,\bar\theta\,H_\mu(x)\Big]. \tag{19}$$

The exact expressions for $F_\mu(x), G_\mu(x)$ and $H_\mu(x)$, in terms of component fields, are

$$\begin{aligned} F_\mu(x) &= e\,\big[\bar\psi\,A_\mu\,\bar b_1 - \mathrm{i}\,\bar\psi\,\partial_\mu\bar C\,\psi - \bar b_2\,A_\mu\,\psi\big],\\ G_\mu(x) &= e\,\big(\bar\psi\,A_\mu\,b_2 - \mathrm{i}\,\bar\psi\,\partial_\mu C\,\psi - b_1\,A_\mu\,\psi\big),\\ H_\mu(x) &= -e\,\big[\bar\psi\,A_\mu\,f + \bar f\,A_\mu\,\psi - \bar\psi\,\partial_\mu\bar C\,b_2 + \bar\psi\,\partial_\mu C\,\bar b_1\\ &\quad + \bar\psi\,\partial_\mu B\,\psi + \mathrm{i}\,\bar b_2\,A_\mu\,b_2 - \mathrm{i}\,b_1\,A_\mu\,\bar b_1\\ &\quad - b_1\,\partial_\mu\bar C\,\psi + \bar b_2\,\partial_\mu C\,\psi\big]. \end{aligned} \tag{20}$$

It should be noted that the above equations (19) and (20) have emerged from the first term of (17). Now, we first set the Grassmannian components (i.e. the coefficients of $d\theta$ and $d\bar\theta$) equal to zero because there are no such types of terms for comparison on the r.h.s. of (11). From equations (14) and (18), we obtain the following terms with $d\theta$:

$$\begin{aligned} &-\mathrm{i}\,d\theta\,\Big(\bar\psi\bar b_1 + e\bar\psi\bar C\psi\Big) + \mathrm{i}\,d\theta\,\bar\theta\\ &\times\Big[\,\bar\psi f - \mathrm{i}b_1\bar b_1 - \mathrm{i}\,e\,\Big(\bar\psi\bar C b_2 - \bar\psi B\psi + b_1\bar C\psi\Big)\Big]\\ &+ d\theta\,\theta\,\Big[\,\bar b_2\bar b_1 + e\Big(\bar\psi\bar C\bar b_1 + \bar b_2\bar C\psi\Big)\Big] + d\theta\,\theta\,\bar\theta\\ &\times\Big[\,\bar b_2 f + \bar f\bar b_1 + e\Big\{\,\bar\psi\bar C f + \bar f\bar C\psi\\ &\quad + \mathrm{i}\,\Big(b_1\bar C\bar b_1 + \bar b_2 B\psi - \bar b_2\bar C b_2 - \bar\psi B\bar b_1\Big)\Big\}\Big]. \end{aligned} \tag{21}$$

Setting equal to zero the coefficients of $d\theta, d\theta(\bar\theta), d\theta(\theta)$ and $d\theta(\theta\bar\theta)$ independently leads to

$$\begin{aligned} &\bar b_1 = -e\bar C\psi, \quad \bar\psi f - \mathrm{i}e\bar\psi\bar C b_2 + \mathrm{i}e\bar\psi B\psi = 0\,,\\ &\bar b_2\bar b_1 + e\,\big(\bar\psi\bar C\bar b_1 + \bar b_2\bar C\psi\big) = 0\,,\\ &\bar b_2 f + \bar f\bar b_1 + e\,\big\{\bar\psi\bar C f + \bar f\bar C\psi + \mathrm{i}\\ &\qquad\times\big(b_1\bar C\bar b_1 + \bar b_2 B\psi - \bar b_2\bar C b_2 - \bar\psi B\bar b_1\big)\big\} = 0\,. \end{aligned} \tag{22}$$

In the second entry, we have used $-\mathrm{i}b_1\bar b_1 - \mathrm{i}eb_1\bar C\psi = 0$ because $\bar b_1 = -e\bar C\psi$. The analogue of (21), which emerges from (14) and (18) with the differential $d\bar\theta$, is as follows:

$$\begin{aligned} &-\mathrm{i}\,d\bar\theta\,\Big(\bar\psi b_2 + e\bar\psi C\psi\Big) - \mathrm{i}\,d\bar\theta\,\theta\\ &\times\Big[\,\bar\psi f + \mathrm{i}\bar b_2 b_2 + \mathrm{i}\,e\,(\bar\psi C\bar b_1 + \bar\psi B\psi + \bar b_2 C\psi)\Big]\\ &+ d\bar\theta\,\bar\theta\,\Big[\,b_1 b_2 + e\Big(\bar\psi C b_2 + b_1 C\psi\Big)\Big] + d\bar\theta\,\theta\,\bar\theta\\ &\times\Big[b_1 f + \bar f b_2 + e\Big\{\bar\psi C f + \bar f C\psi + \mathrm{i}\\ &\qquad\times\Big(b_1 C\bar b_1 + b_1 B\psi - \bar b_2 C b_2 - \bar\psi B b_2\Big)\Big\}\Big]. \end{aligned} \tag{23}$$

For $\bar\psi \neq 0$, we obtain the following independent relations from the above equation:

$$\begin{aligned}
& b_2 = -eC\psi\,, \\
& \bar\psi f + \mathrm{i}e\bar\psi C\bar b_1 + \mathrm{i}e\bar\psi B\psi = 0\,, \\
& b_1 b_2 + e\Big(\bar\psi C b_2 + b_1 C\psi\Big) = 0\,, \\
& b_1 f + \bar f b_2 + e\Big\{\bar\psi C f + \bar f C\psi + \mathrm{i} \\
& \qquad \times \Big(b_1 C\bar b_1 + b_1 B\psi - \bar b_2 C b_2 - \bar\psi B b_2\Big)\Big\} = 0\,.
\end{aligned} \tag{24}$$

In the above, in the second entry, $\mathrm{i}\bar b_2 b_2 + \mathrm{i}e\bar b_2 C\psi = 0$ has been exploited due to the fact that $b_2 = -eC\psi$. It can be readily checked that the equations (22) and (24) allow the following expression for f as the solution to the second entries of both of them:

$$f = -\mathrm{i}\,e\,\left(B + e\,\bar C\,C\right)\psi\,. \tag{25}$$

The substitution of all the above values for $\bar b_1, b_2$ and f in (12) yields the following expansion of the superfield Ψ in the language of the (anti-)BRST transformations (2):

$$\begin{aligned}
& \Psi(x,\theta,\bar\theta) \\
& \quad = \psi(x) + \theta\,(s_{ab}\psi(x)) + \bar\theta\,(s_b\psi(x)) + \theta\,\bar\theta\,(s_b s_{ab}\psi(x))\,.
\end{aligned} \tag{26}$$

It will be noted that, so far, the third and fourth entries of (22) and (24) have not been exploited. We shall comment on them a little later (see (33)).

Now the stage is set for the discussion of the coefficients of dx^μ that emerge from (15) and (19). It is straightforward to check that the coefficients of the pure dx^μ from the l.h.s. and r.h.s. do match. Furthermore, the coefficient of $dx^\mu\,\theta$ ought to be zero because there is no such term on the r.h.s. The exact expression for such an equality is as follows:

$$\begin{aligned}
& e\,\Big(\bar\psi\,A_\mu\,\bar b_1 - \mathrm{i}\,\bar\psi\,\partial_\mu\bar C\,\psi - \bar b_2\,A_\mu\,\psi\Big) \\
& \quad + \mathrm{i}\,\Big(\bar b_2\,\partial_\mu\psi - \bar\psi\,\partial_\mu\bar b_1\Big) = 0\,.
\end{aligned} \tag{27}$$

Exploiting the values of $\bar b_1$ and b_2 from (22) and (24), the above equation leads to the following useful equation for the unknown local parameter field $\bar b_2(x)$:

$$\mathrm{i}\,\Big(\,\bar b_2 + e\,\bar\psi\,\bar C\,\Big)\,\Big(\,\partial_\mu\psi + \mathrm{i}eA_\mu\psi\,\Big) = 0\,. \tag{28}$$

This yields the value for $\bar b_2$ to be $-e\bar\psi\bar C$ (i.e. $\bar b_2 = -e\bar\psi\bar C$). It is clear that $D_\mu\psi = \partial_\mu\psi + \mathrm{i}eA_\mu\psi \neq 0$ for an interacting $U(1)$ gauge theory because the interaction term $-e\bar\psi\gamma^\mu A_\mu\psi$ is hidden in the covariant derivative, in the sense that it emerges from $\mathrm{i}\bar\psi\gamma^\mu D_\mu\psi$. Setting equal to zero the coefficient of $dx^\mu\,\bar\theta$, we obtain

$$\begin{aligned}
& e\,\Big(\bar\psi\,A_\mu\,b_2 - \mathrm{i}\,\bar\psi\,\partial_\mu C\,\psi - b_1\,A_\mu\,\psi\Big) \\
& \qquad + \mathrm{i}\,\Big(b_1\,\partial_\mu\psi - \bar\psi\,\partial_\mu b_2\Big) = 0\,.
\end{aligned} \tag{29}$$

Substituting the value of b_2 (i.e. $b_2 = -e\,C\,\psi$), we obtain the following relation for an unknown local parameter component field $b_1(x)$ of the expansion (12):

$$\mathrm{i}\,\Big(\,b_1 + e\,\bar\psi\,C\,\Big)\,\Big(\,\partial_\mu\psi + \mathrm{i}eA_\mu\psi\,\Big) = 0\,. \tag{30}$$

The above equation produces the value of b_1 as $-e\,\bar\psi\,C$ in a unique fashion. Ultimately, we now focus on the computation of the coefficient of $dx^\mu\,\theta\,\bar\theta$, which will naturally be set equal to zero because there is no such term on the r.h.s. Mathematically, the precise expression, for the above statement of equality, is as follows:

$$\begin{aligned}
& -e\,\Big[\,\bar\psi\,A_\mu\,f + \bar f\,A_\mu\,\psi - \bar\psi\,\partial_\mu\bar C\,b_2 + \bar\psi\,\partial_\mu C\,\bar b_1 + \bar\psi\,\partial_\mu B\,\psi \\
& + \mathrm{i}\,\bar b_2\,A_\mu\,b_2 - \mathrm{i}\,b_1\,A_\mu\,\bar b_1 - b_1\,\partial_\mu\bar C\,\psi + \bar b_2\,\partial_\mu C\,\psi\,\Big] \\
& + \mathrm{i}\,\Big[\,\bar\psi\,\partial_\mu f + \bar f\,\partial_\mu\psi + \mathrm{i}\,\left(\bar b_2\,\partial_\mu b_2 - b_1\,\partial_\mu\bar b_1\right)\,\Big] = 0\,.
\end{aligned} \tag{31}$$

Substituting the values $b_1 = -e\bar\psi C$, $\bar b_1 = -e\bar C\psi$, $b_2 = -eC\psi$, $\bar b_2 = -e\bar\psi\bar C$, $f = -\mathrm{i}e(B + e\bar C C)\,\psi$, we obtain the following relationship for the unknown local parameter field $\bar f(x)$:

$$\mathrm{i}\,\left(\bar f - \mathrm{i}\,e\,\bar\psi\,B + \mathrm{i}\,e^2\,\bar\psi\,\bar C\,C\right)\,\left(\,\partial_\mu\psi + \mathrm{i}eA_\mu\psi\,\right) = 0\,. \tag{32}$$

The above relation yields the expression for $\bar f(x)$ in terms of the fields of the (anti-)BRST invariant Lagrangian density (1). Together, all the local (unknown) secondary component fields, in the expansion (12) of the superfields $(\Psi, \bar\Psi)(x,\theta,\bar\theta)$, are as follows:

$$\begin{aligned}
& b_1 = -e\bar\psi C\,, && b_2 = -eC\psi\,, \\
& \bar b_1 = -e\bar C\psi\,, && \bar b_2 = -e\bar\psi\bar C\,, \\
& f = -\mathrm{i}e\,[\,B + e\bar C C\,]\,\psi\,, && \bar f = +\mathrm{i}e\,\bar\psi\,[\,B + eC\bar C\,]\,.
\end{aligned} \tag{33}$$

It is worthwhile to mention that exactly the same expressions as quoted above were obtained in our earlier work [11], where the invariance of the conserved matter current on the supermanifold was imposed. However, the above solutions in [11] were not mathematically unique. In our present endeavor, we have been able to show the uniqueness and exactness of the solutions (33). Furthermore, the solutions (33) do satisfy all the conditions of (22) and (24) which have appeared as the third and fourth entries. With the values from (33), the super expansion of the superfield $\bar\Psi(x,\theta,\bar\theta)$, in the language of the (anti-)BRST transformations (2), is as illustrated below:

$$\begin{aligned}
& \bar\Psi(x,\theta,\bar\theta) \\
& \quad = \bar\psi(x) + \theta\,\left(s_{ab}\bar\psi(x)\right) + \bar\theta\,\left(s_b\bar\psi(x)\right) + \theta\,\bar\theta\,\left(s_b s_{ab}\bar\psi(x)\right)\,.
\end{aligned} \tag{34}$$

The above expansion, in terms of $s_{(a)b}$, bears exactly the same appearance as its counterpart in (26) where the expansion for the superfield $\Psi(x,\theta,\bar\theta)$ has been given.

5 Conclusions

The long-standing problem of the derivation of the nilpotent (anti-)BRST symmetry transformations for the matter (e.g. Dirac) fields of an interacting gauge theory (e.g. QED), in the framework of the superfield formalism[6], has been resolved uniquely in our present endeavor. We have invoked an additional gauge invariant restriction (cf. (11)), besides the usual horizontality condition (cf. (5)), on the supermanifold to obtain the off-shell nilpotent symmetry transformations for the Dirac fields of QED.

It is very interesting and gratifying that both the restrictions on the supermanifold are complementary and consistent with each other and, more importantly, they are intertwined in the sense that they owe their origin to the nilpotent (super) exterior derivatives $(\tilde{d})d$ and 1-form (super) connection $(\tilde{A}^{(1)})A^{(1)}$. The present extended version of the usual superfield formalism, which leads to the derivation of mathematically exact expressions for the off-shell nilpotent (anti-)BRST symmetry transformations associated with all the fields of QED, has been christened as the augmented superfield approach to the BRST formalism.

It is worthwhile to note that the horizontality condition (cf. (5)) on the supermanifold (that leads to the derivation of exact nilpotent symmetry transformations for the gauge and (anti-)ghost fields) is precisely a gauge covariant statement because, for the non-Abelian $SU(N)$ gauge theory, the 2-form curvature $F^{(2)}$ transforms as $F^{(2)} \to (F^{(2)})' = UF^{(2)}U^{-1}$ (with $U \in SU(N)$). It is another matter that it becomes a gauge invariant statement (i.e. $F^{(2)} \to (F^{(2)})' = UF^{(2)}U^{-1} = F^{(2)}$) for our present case of an interacting Abelian $U(1)$ gauge theory. In contrast, the additional restriction (11), invoked on the supermanifold, is primarily a gauge invariant statement. In fact, its gauge covariant version on the supermanifold leads to absurd results (even for the simplest case of an interacting Abelian $U(1)$ gauge theory), as can be seen explicitly in the appendix.

Our present theoretical arsenal of the augmented superfield formalism has already been exploited [18, 19] for the derivation of the exact and unique nilpotent symmetry transformations for (i) the complex scalar fields in interaction with the $U(1)$ gauge field (see e.g. [11, 12] for earlier works) and (ii) the Dirac fields in interaction with the $SU(N)$ non-Abelian gauge field (see e.g. [14] for our earlier work). As is evident from our discussions, $B_\mu, \mathcal{F}, \bar{\mathcal{F}}$ form the vector multiplet of the 1-form superfield $\tilde{A}^{(1)} = dZ^M \tilde{A}_M$. One of the most intriguing questions, in this context, is to find some multiplet of a superfield that can accommodate the spinor superfields $\Psi(x, \theta, \bar{\theta})$ and $\bar{\Psi}(x, \theta, \bar{\theta})$. So far, we have not been able to find this multiplet. It would be interesting to find the answer to this question. These are some of the immediate and urgent issues that are under investigation at the moment and our results will be reported in future publications [20].

[6] In the known literature on the usual superfield formulation, only the nilpotent BRST-type symmetry transformations for the gauge and (anti-)ghost fields have been derived without any comment on such types of transformations associated with the matter fields of an interacting gauge theory [1–5, 10]. However, in our recent works on the augmented superfield formalism [11–16], this problem has been addressed.

Acknowledgements. Fruitful conversations with L. Bonora (SISSA, Italy), K.S. Narain (AS-ICTP, Italy) and M. Tonin (Padova, Italy) are gratefully acknowledged. The warm hospitality extended at the AS-ICTP and the Physics Department of Padova University is also thankfully acknowledged.

Appendix : A

Let us begin with the gauge *covariant* version of (11), namely

$$\left(\tilde{d} + \mathrm{i}e\tilde{A}^{(1)}_{(h)}\right) \Psi\left(x, \theta, \bar{\theta}\right) = \left(d + \mathrm{i}eA^{(1)}\right) \psi(x) , \qquad \text{(A.1)}$$

where the symbols carry their usual meaning, as discussed earlier. It is clear that the r.h.s. of the above equation (i.e. $dx^\mu(\partial_\mu + \mathrm{i}eA_\mu)\psi(x)$) contains a single differential dx^μ. However, the l.h.s. would yield the coefficients of the differentials dx^μ, $d\theta$ and $d\bar{\theta}$. In fact, the l.h.s. consists of $\tilde{d}\Psi$ and $\mathrm{i}e\tilde{A}^{(1)}_{(h)}\Psi$. The former term can be written as

$$\begin{aligned} \tilde{d}\,\Psi = dx^\mu & \left(\partial_\mu\psi + \mathrm{i}\theta\partial_\mu\bar{b}_1 + \mathrm{i}\bar{\theta}\partial_\mu b_2 + \mathrm{i}\theta\bar{\theta}\partial_\mu f\right) \\ & + \mathrm{i}d\theta\left(\bar{b}_1 + \bar{\theta}f\right) + \mathrm{i}d\bar{\theta}\left(b_2 - \theta f\right) , \end{aligned} \qquad \text{(A.2)}$$

and the latter term can be explicitly expressed as

$$\begin{aligned} & \mathrm{i}e\tilde{A}^{(1)}_{(h)}\Psi(x, \theta, \bar{\theta}) \\ & \quad = \mathrm{i}e\left[\, dx^\mu B^{(h)}_\mu + d\theta\bar{\mathcal{F}}^{(h)} + d\bar{\theta}\mathcal{F}^{(h)} \,\right] \Psi(x, \theta, \bar{\theta}) . \end{aligned} \qquad \text{(A.3)}$$

The latter two terms of the above expression yield the following coefficients of $d\theta$ and $d\bar{\theta}$:

$$\begin{aligned} & \mathrm{i}ed\theta\left(\bar{C}\psi\right) + ed\theta(\theta)\left(\bar{C}\bar{b}_1\right) + ed\theta\left(\bar{\theta}\right)\left[\bar{C}b_2 - B\psi\right] \\ & - ed\theta\left(\theta\bar{\theta}\right)\left[\bar{C}f - \mathrm{i}B\bar{b}_1\right] , \end{aligned} \qquad \text{(A.4)}$$

$$\begin{aligned} & \mathrm{i}ed\bar{\theta}(C\psi) + ed\bar{\theta}(\theta)\left[C\bar{b}_1 + B\psi\right] + ed\bar{\theta}\left(\bar{\theta}\right)(Cb_2) \\ & + ed\bar{\theta}(\theta\bar{\theta})[Cf - \mathrm{i}Bb_2] . \end{aligned} \qquad \text{(A.5)}$$

It is straightforward to note that the above coefficients would not emerge from the r.h.s. Thus, these coefficients would be set equal to zero. Equating the coefficients of $d\theta$, $d\theta(\theta)$, $d\theta(\bar{\theta})$ and $d\theta(\theta\bar{\theta})$ equal to zero, we obtain the following conditions:

$$\begin{aligned} & \bar{b}_1 = -e\bar{C}\psi , && \bar{C}\bar{b}_1 = 0 , \\ & f = -\mathrm{i}e\left(B\psi - \bar{C}b_2\right) , && \bar{C}f - \mathrm{i}B\bar{b}_1 = 0 . \end{aligned} \qquad \text{(A.6)}$$

It should be noted that, in the above computation, exactly similar types of terms have been collected from (A.2) and (A.4). It is obvious that the second and fourth conditions are satisfied if we take into account the value of $\bar{b}_1, f$ and exploit the condition $\bar{C}^2 = 0$. Similarly, we collect the terms of similar kinds from (A.2) and (A.5) and set the coefficients of $d\bar\theta, d\bar\theta(\theta), d\bar\theta(\bar\theta)$ and $d\bar\theta(\theta\bar\theta)$ equal to zero separately and independently. These lead to the following conditions:

$$b_2 = -eC\psi\,, \qquad f = -\mathrm{i}e\left(B\psi + C\bar b_1\right)\,, \qquad Cb_2 = 0\,, \qquad Cf - \mathrm{i}Bb_2 = 0\,. \tag{A.7}$$

It is evident that, in the above, the third and fourth conditions are satisfied. Finally, exploiting the conditions in (A.6) and (A.7), we obtain the following:

$$\bar b_1 = -e\bar C\psi\,, \qquad b_2 = -eC\psi\,, \qquad f = -\mathrm{i}e(B + e\bar C C)\psi\,. \tag{A.8}$$

It is worth emphasizing that the above results are also obtained from the gauge invariant condition (11) when we set equal to zero the coefficients of $d\theta$ and $d\bar\theta$. The key difference between the gauge invariant condition (11) and the gauge covariant condition (A.1) is found to be contained in the coefficients of dx^μ. To make this statement more transparent, we expand the first term (i.e. $\mathrm{i}e\,dx^\mu\, B^{(h)}_\mu$) of (A.3) as follows:

$$\mathrm{i}e dx^\mu \Big[A_\mu\psi + \mathrm{i}\theta\left(A_\mu\bar b_1 - \mathrm{i}\partial_\mu\bar C\psi\right) + \mathrm{i}\bar\theta(A_\mu b_2 - \mathrm{i}\partial_\mu C\psi) + \mathrm{i}\theta\bar\theta Q_\mu\Big]\,, \tag{A.9}$$

where the explicit expression for the quantity Q_μ is

$$Q_\mu = A_\mu f + \partial_\mu B\psi + \partial_\mu C\bar b_1 - \partial_\mu\bar C b_2\,. \tag{A.10}$$

A careful observation of the equations (A.2) and (A.9) demonstrates that there are coefficients of dx^μ, $dx^\mu(\theta)$, $dx^\mu(\bar\theta)$ and $dx^\mu(\theta\bar\theta)$. It is straightforward to note that the coefficient of pure dx^μ from the l.h.s. does match with the one that emerges from the r.h.s. The coefficients of $dx^\mu(\theta)$, $dx^\mu(\bar\theta)$ and $dx^\mu(\theta\bar\theta)$ are listed below:

$$\begin{aligned} &dx^\mu\,(\theta)\left[\,\mathrm{i}\partial_\mu\bar b_1 - eA_\mu\bar b_1 + \mathrm{i}e\partial_\mu\bar C\psi\,\right],\\ &dx^\mu\,(\bar\theta)\left[\,\mathrm{i}\partial_\mu b_2 - eA_\mu b_2 + \mathrm{i}e\partial_\mu C\psi\,\right],\\ &dx^\mu(\theta\bar\theta)\left[\,\mathrm{i}\partial_\mu f - eA_\mu f + e\partial_\mu\bar C b_2 - e\partial_\mu C\bar b_1 - e\partial_\mu B\psi\,\right]. \end{aligned} \tag{A.11}$$

As is evident, these coefficients are set to zero to have conformity with the gauge covariant condition in (A.1). The substitution of the values from (A.8) into the above conditions leads to the following restrictions:

$$-\mathrm{i}e\bar C D_\mu\psi = 0\,, \qquad -\mathrm{i}eCD_\mu\psi = 0\,, \qquad e(B - eC\bar C)D_\mu\psi = 0\,. \tag{A.12}$$

The above restrictions do not lead to any physically interesting solutions because they imply that $D_\mu\psi = 0$ for $C \neq 0, \bar C \neq 0, B \neq eC\bar C$. However, for an interacting Abelian $U(1)$ gauge theory, this condition is absurd. The other choices, for instance, the conditions: $C = 0, \bar C = 0$ and $B = eC\bar C$ (for $D_\mu\psi \neq 0$), are also not acceptable. Thus, we conclude that the gauge covariant condition (A.1) does not lead to exact derivations of the nilpotent symmetry transformations for the matter fields in QED. In contrast, the gauge invariant restriction (11) does lead to exact derivations.

References

1. J. Thierry-Mieg, J. Math. Phys. **21**, 2834 (1980); Nuovo Cimento A **56**, 396 (1980)
2. M. Quiros, F.J. De Urries, J. Hoyos, M.L. Mazon, E. Rodrigues, J. Math. Phys. **22**, 767 (1981)
3. R. Delbourgo, P.D. Jarvis, J. Phys. A: Math. Gen. **15**, 611 (1981); R. Delbourgo, P.D. Jarvis, G. Thompson, Phys. Lett. B **109**, 25 (1982)
4. L. Bonora, M. Tonin, Phys. Lett. B **98**, 48 (1981); L. Bonora, P. Pasti, M. Tonin, Nuovo Cimento A **63**, 353 (1981)
5. L. Baulieu, J. Thierry-Mieg, Nucl. Phys. B **197**, 477 (1982); Nucl. Phys. B **228**, 259 (1982); L. Alvarez-Gaumé, L. Baulieu, Nucl. Phys. B **212**, 255 (1983)
6. N. Nakanishi, I. Ojima, Covariant Operator Formalism of Gauge Theories and Quantum Gravity (World Scientific, Singapore, 1990)
7. K. Nishijima, in Progress in Quantum Field Theory, ed. by H. Ezawa, S. Kamefuchi (North-Holland, Amsterdam, 1986) p 99; K. Nishijima, Czech. J. Phys. **46**, 1 (1996)
8. M. Henneaux, C. Teitelboim, Quantization of Gauge Systems (Princeton University Press, Princeton, 1992)
9. S. Weinberg, The Quantum Theory of Fields: Modern Applications, vol. 2 (Cambridge University Press, Cambridge, 1996)
10. D.S. Hwang, C.-Y. Lee, J. Math. Phys. **38**, 30 (1997)
11. R.P. Malik, Phys. Lett. B **584**, 210 (2004) [hep-th/0311001]
12. R.P. Malik, J. Phys. A: Math. Gen. **37**, 5261 (2004) [hep-th/0311193]
13. R.P. Malik, Int. J. Geom. Methods Mod. Phys. **1**, 467 (2004) [hep-th/0403230]
14. R.P. Malik, Int. J. Mod. Phys. A **20**, 4899 (2005) [hep-th/0402005]
15. R.P. Malik, Mod. Phys. Lett. A **20**, 1767 (2005) [hep-th/0402123]
16. R.P. Malik, Eur. Phys. J. C **45**, 513 (2006)
17. I.J.R. Aitchison, A.J.G. Hey, Gauge Theories in Particle Physics: A Practical Introduction (Adam Hilger, Bristol, 1982)
18. R.P. Malik, hep-th/0510226
19. R.P. Malik, B.P. Mandal, hep-th/0512334
20. See, e.g., R.P. Malik, hep-th/0603049